2^{nd} Course
Beijing
08-11-2019

Recall from last time

$$\begin{tikzcd} & {[E:\mathbb{Q}_p]<\infty} \\ E \arrow[ur] \arrow[dr] & \\ & E=\mathbb{F}_q((\pi)) \end{tikzcd}$$

$\mathcal{O}_E/\pi=\mathbb{F}_q$

"p-adic Hodge theory wrt. coeff. in E"

"Hartl-Pink - equal char. local Shtukas"

$F|\mathbb{F}_q$ perfectoid field $|\cdot|:F\to\mathbb{R}_+$

$$A=\begin{cases} W_{\mathcal{O}_E}(\mathcal{O}_F)=\left\{\sum_{n\geq 0}[x_n]\pi^n \;\middle|\; x_n\in\mathcal{O}_F\right\} \\ \mathcal{O}_F[\![\pi]\!] \end{cases}$$

$+\,(\pi,[\varpi])$-adic topology $\qquad 0<|\varpi|<1$

$$\begin{tikzcd} Y=\mathrm{Spa}(A,A)\setminus V(\pi[\varpi]) \arrow[d] \\ \mathrm{Spa}(E) \end{tikzcd}$$

"Stein" E-adic space

E-Frechet algebra $\mathcal{O}(Y)$:

$$\mathbb{A}\left[\frac{1}{\pi}, \frac{1}{[\varpi]}\right] = \left\{ \sum_{n>>-\infty} [x_n]\pi^n \;/\; x_n \in F,\ \sup_n |x_n| < +\infty \right\}$$

$\rho \in]0,1[$ $\quad |f|_\rho = \sup_n |x_n|\rho^n$ **Gauss norm**

radius

$\rho = q^{-r}$, $r \in]0,+\infty[$, $|\cdot|_\rho = q^{-v_r(\cdot)}$

$$v_r(f) = \inf_{n\in\mathbb{Z}} \{ v(x_n) + nr \}$$

$$\left[B := \mathcal{O}(Y) = \text{Completion of } \mathbb{A}\left[\frac{1}{\pi}, \frac{1}{[\varpi]}\right] \text{ w.r.t. } (|\cdot|_\rho)_{\rho\in]0,1[} \right]$$

$I \subset]0,1[$, $B_I =$ " " $(|\cdot|_\rho)_{\rho\in I}$

I Compact $\Rightarrow B_I =$ Completion w.r.t. $\sup\{|\cdot|_{\rho_1}, |\cdot|_{\rho_2}\}$

$\overset{\shortparallel}{[\rho_1, \rho_2]}$ (maximum modulus principle)

Case $E = \mathbb{F}_q((\pi))$: $Y = \mathbb{D}_F^* = \{0 < |\pi| < 1\} \subset \mathbb{A}_F^1$

$$B = \left\{ \sum_{n \in \mathbb{Z}} x_n \pi^n \,\Big/\, x_n \in F \;\; \forall \rho \in]0,1[\;\; \lim_{|n| \to +\infty} |x_n| \rho^n = 0 \right\}$$

$$\cup$$

$$\mathbb{F}_q((\pi))$$

$$\begin{tikzcd} \mathbb{D}_F^* \arrow[d, "\text{not loc. of f.t.}"] \\ \mathrm{Spa}(\mathbb{F}_q((\pi))) \end{tikzcd}$$

⚠ : $E/\mathbb{Q}_q$ $(x_n)_{n \in \mathbb{Z}} \in F^{\mathbb{Z}}$ satisfying $\forall \rho \in]0,1[$

$$\lim_{|n| \to +\infty} |x_n| \rho^n = 0.$$

$\Rightarrow \sum_{n \in \mathbb{Z}} [x_n] \pi^n \in B$ well defined (c.v.)

but à priori :
- * not every element of B is of this form i.e. "has a Laurent expansion around $\pi = 0$"
- * the sum or product of two elements of this form may not be of this form.

Multiplicative property of Gauss norms

Prop: $|fg|_\rho = |f|_\rho \cdot |g|_\rho$ i.e. $|\cdot|_\rho$ = absolute value

or $\boxed{v_r \text{ is a valuation.}}$

→ proof not such difficult

If $\rho \in I$ compact $|\cdot|_\rho \in \mathcal{M}(B_I)$ Berkovich spectrum

Want to do some holomorphic analysis
→ locate zeroes of hol. fct.

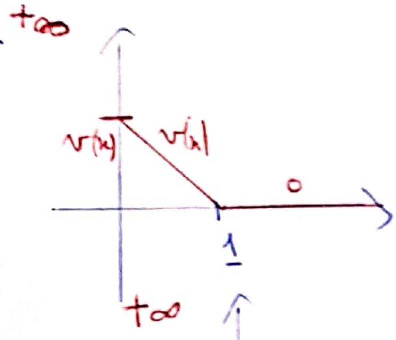
Newton polygon: $E = \mathbb{F}_q((\pi))$.

$$\begin{array}{l} f \in \mathcal{O}(\mathbb{D}^*_F) \\ \quad \| \\ \displaystyle\sum_{n\in\mathbb{Z}} x_n \pi^n \end{array}$$

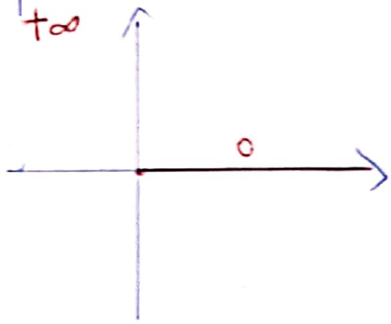
Can define $\mathrm{Newt}(f) : \mathbb{R} \longrightarrow]-\infty, +\infty]$

$$\begin{array}{l} \qquad \| \\ \text{decreasing convex hull of } \{(n, v(x_n))\}_{n\in\mathbb{Z}} \\ \| \\ \text{polygon breakpoint of } x\text{-coordinate} \in \mathbb{Z} \end{array}$$

Ex: $\mathrm{Newt}(\pi - [x]) = \Bigg\{$



if $v(x) \geq 0$



if $v(x) \leq 0$

>0 slopes of $\mathrm{Newt}(f) = v(\text{roots of } f)$ in F with multiplicities $\in \,]0, +\infty[$

→ See Fargues and "Vector bundles and p-adic ~~Galois~~ Hodge theory ~~representations~~"

Case $E|\mathbb{Q}_p$: Problem $f \in B$ how to define its Newton polygon?

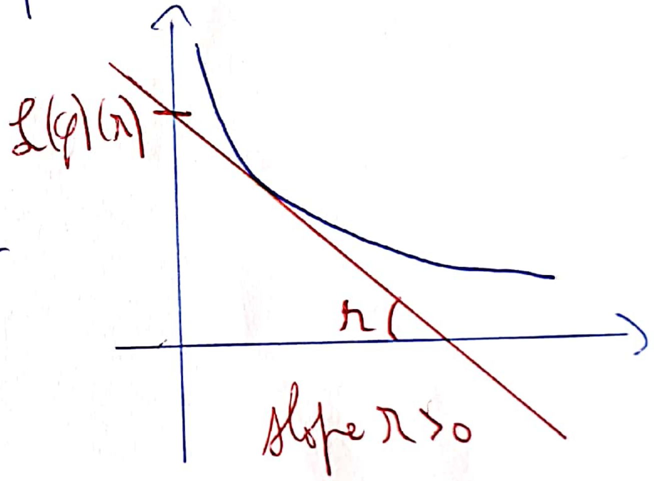
No expansion $\sum_{n \in \mathbb{Z}} [x_n] \pi^n$ ....

Solution: Legendre transform

Recall:

$$\mathcal{L} : \left\{ \begin{array}{l} \text{Convex decreasing} \\ \text{fct. : } \mathbb{R} \to \mathbb{R}\cup\{+\infty\} \\ \qquad \not\equiv +\infty \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{Concave functions} \\ ]0,+\infty[\to \mathbb{R}\cup\{-\infty\} \\ \qquad \not\equiv -\infty \end{array} \right\}$$

$$\left\{ \begin{array}{l} \mathcal{L}(\varphi)(r) = \inf\limits_{t\in\mathbb{R}} \varphi(t) + tr \\ \mathcal{L}^{-1}(\psi)(t) = \sup\limits_{r\in]0,+\infty[} \psi(r) - rt \end{array} \right.$$



Tropical analog of Fourier transform

$$(\mathbb{R}, +, \times) \longleftrightarrow (\mathbb{R}, \inf, +)$$

Tropical Convolution: f, g convex

$$(f \circledast g)(h) = \inf_{a+b=h} \left(f(a) + g(b) \right)$$

$$\mathcal{L}(f \circledast g) = \mathcal{L}(f) + \mathcal{L}(g)$$

Moreover: $\mathcal{L} : \{\text{polygons}\} \xrightarrow{\sim} \{\text{polygons}\}$

polygon := piecewise affine

duality: slopes $\longleftrightarrow$ x- Coordinnate of breakpoint

i.e. slopes of $\mathcal{L}(\varphi)$ = x- Coordinnates of breakpoints of φ

x- Coordinnate of breakpoints of φ = slopes of $\mathcal{L}(\varphi)$

Prop: $f \in B \setminus \{0\}$ $f_n \xrightarrow[n \to +\infty]{} f$, $f_n \in \mathbb{A}[\frac{1}{u}, \frac{1}{[\varpi]}]$

Then $\forall K$ Compact $\subset]0, +\infty[$ $\exists N$, $n \geq N \Rightarrow \forall r \in K, v_r(f) = v_r(f_n$

$\longrightarrow$ **Dini:** $\varphi_n :]0, +\infty[\longrightarrow \mathbb{R}$ sequence of Concave functions that C.V. simply toward φ. Then the C.V. is uniform on any Compact of $]0, +\infty[$

Corollary: $\forall f \in B \setminus \{0\}$ $\forall K$ Compact $\subset]0, +\infty[$

$\exists g \in \mathbb{A}[\frac{1}{u}, \frac{1}{[\varpi]}]$ s.t. $\forall r \in K$, $v_r(f) = v_r(g)$

$\Rightarrow$ [The concave function $\begin{cases}]0,+\infty[\longrightarrow \mathbb{R} \\ r \longmapsto v_r(x) \end{cases}$

is a polygon with integral slopes.]

Def. For $f \in B$ we note $\mathrm{Newt}(f)$ for the inverse Legendre transform of $r \mapsto v_r(f)$

= polygon with integral x-coordinate breakpoints

Prop. $\mathrm{Newt}(fg) = \mathrm{Newt}(f) \circledast \mathrm{Newt}(g)$

$\rightarrow$ Consequence of $v_r(fg) = v_r(f) + v_r(g)$

$\Rightarrow$ Slopes of $\mathrm{Newt}(fg)$ obtained by concatenation of the one of $\mathrm{Newt}(f)$ and $\mathrm{Newt}(g)$.

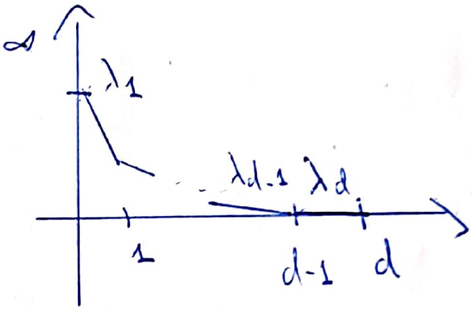
Rem: Of course if $f = \sum_{n \gg -\infty} [x_n]\pi^n \in A[\frac{1}{\pi}, \frac{1}{[\varpi]}]$

then $\mathrm{Newt}(f) =$ decreasing convex hull of

$$\{(n, v(x_n))\}_{n \in \mathbb{Z}}$$

Ex. $\lambda_1 = v(a_1) \geq \dots \geq v(a_d) = \lambda_d > 0$

$$\mathrm{Newt}\big((\pi - [a_1]) \dots (\pi - [a_d])\big) =$$



→ "unprovable" using the multiplicative structure of Witt vectors. Very easy using Legendre transforms.

Newton polygon of elements of B_I, $I \subset]0,1[$ compact

Problem: $I \subset]0,1[$ compact, $f \in B_I$

Would like to define $\mathrm{Newt}_I(f) =$ polygon whose slopes $\in -\log_q(I) \subset]0,+\infty[$

Such that if $f \in \mathbb{A}[\frac{1}{\pi}, \frac{1}{[\varpi]}]$ then

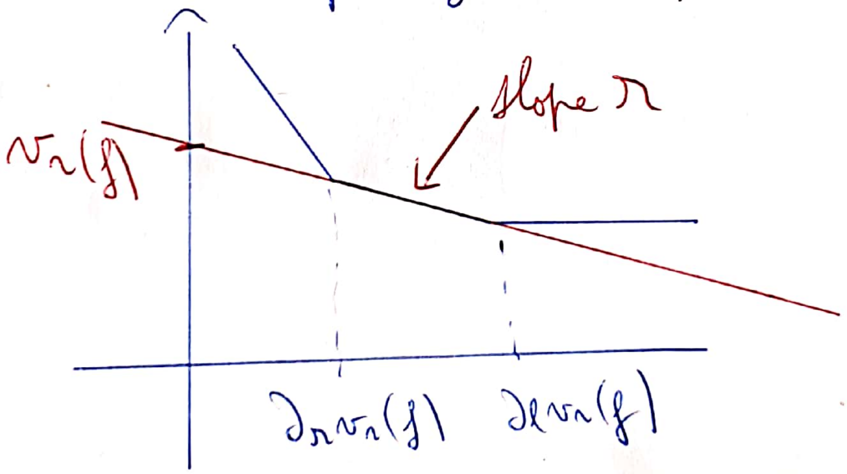
$$\mathrm{Newt}_I(f) = \mathrm{Newt}(f) \setminus \{\text{slope } \lambda \text{ s.t. } q^{-\lambda} \notin I\}$$

If $I = \{q^{-r}\}$ and $f \in \mathbb{A}[\frac{1}{\pi}, \frac{1}{[\varpi]}]$ then $v_r(f)$ does not determine completely $\mathrm{Newt}_I(f)$.

Solution: But $(v_r(f), \partial_\ell v_r(f), \partial_r v_r(f))$ determine it completely

left derivative

right derivative



→ recall: slopes of $r \mapsto v_r(f)$ are the x-coordinate of breakpoints.

The rank 2 valuations

$$f \longmapsto \begin{cases} (v_r(f), \partial_r v_r(f)) \\ (v_r(f), -\partial_\ell v_r(f)) \end{cases}$$

are specializations of v_n and thus can be extended to B_I when $q^{-n} \in I$.

$\Rightarrow \forall f \in B_I \setminus \{0\}$ if $q^{-n} \in I$ we can define $\partial_{\ell} v_n(f)$ and $\partial_{r} v_n(f)$ $\in \mathbb{Z}$.

Using this one can define $\mathrm{Newt}_I(f)$ for $f \in B_I$.

We still have $\mathrm{Newt}_I(fg) =$ Concatenation of $\mathrm{Newt}_I(f)$ and $\mathrm{Newt}_I(g)$

Zeros of holomorphic functions

Jensen's inequality / equality:

Recall: $f \in \mathcal{O}(\mathbb{C}) \setminus \{0\}$ s.t. $f(0) \neq 0$

$R > 0$ such that f has no zero on $\{|z| = R\}$

$a_1, \ldots, a_n$ zeros of f in $\{|z| < R\}$

Then (Jensen):

$$\ln|f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \ln|f(Re^{i\theta})|\, d\theta - n \ln R + \sum_{i=1}^{n} \ln|a_i|$$

(Write that log|f| is harmonic
+ mean value principle)

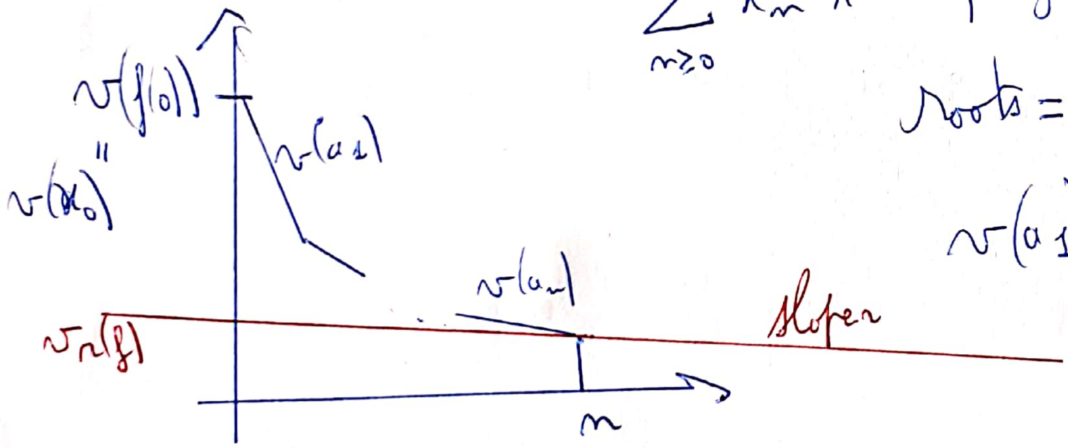
⇒ Inequality $\left[\ln|f(0)| \leq M(R) - n\ln R + \sum_{i=1}^{n} \ln|a_i|\right]$

In the non-archimedean setting there is an equality. ($f \in \mathcal{O}(\mathbb{D}_F)$) $E = \mathbb{F}_q((\pi))$

$$\overset{\shortparallel}{\sum_{n\geq 0} x_n \pi^n}, \quad f(0) = x_0 \neq 0$$

roots $= (a_i)_{i\geq 1}$

$v(a_1) \geq \dots \geq v(a_n) \geq \dots$



slopes of Newt(f) = v(roots of f)

$\Leftrightarrow v(f(0)) = v_r(f) - nr + \sum_{i=1}^{n} v(a_i)$

We want to do the same for $E = \mathbb{Q}_p$.

→ Need to define zeros of f in this setting.

For $E = \mathbb{F}_q((\pi))$ the zeros are elements of $\underbrace{|\mathbb{D}_F^*|}^{cl}$

classical Take points of rigid space $\mathbb{D}_F^*$

$$|\mathbb{D}_F^*|^{cl} = \{ z \in \overline{F} \,/\, 0 < |z| < 1 \} \,/\, Gal(\overline{F}|F)$$

$$= \{ P \in \mathcal{O}_F[\pi] \text{ unitary irreducible satisfying } 0 < |P(0)| < 1 \}$$

$$= \{ f \in A \text{ primitive irreducible} \} / A^\times$$

↑ Weierstrass factorization

or distinguished ↓

where $f = \sum_{n \geq 0} x_n \pi^n \in \mathcal{O}_F[[\pi]]$ is primitive of deg. d if $x_0 \neq 0$, $x_0, \dots, x_{d-1} \in \mathfrak{m}_F$ and $x_d \in \mathcal{O}_F^\times$

We are going to use this point of view to define $|Y_F|^{cl}$ when $E|\mathbb{Q}_p$.